Remarks on the neutrino oscillation formula

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Abstract

We show that the neutrino oscillation formula recently derived in the quantum field theory framework holds true despite the arbitrariness in the mass parameter for the flavor fields. This formula is exact and exhibits new features with respect to the usual Pontecorvo formula, which is however valid in the relativistic limit.

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I. INTRODUCTION

In view of the great theoretical and experimental interest [1] several papers have been recently devoted to the quantum field theoretical approach to the neutrino mixing and oscillations [2–5]. In particular, the study of the generator of the Pontecorvo mixing transformations [6] has shown [3] that the Hilbert space where the mixed (flavor) field operators are defined is unitarily inequivalent, in the infinite volume limit, to the Hilbert space for the original (unmixed) field operators. Such a finding leads to a novel understanding of the field mixing and to a new, exact oscillation formula [4], which reduces to the Pontecorvo one in the relativistic limit.

As a matter of fact, the problem of the definition and of the physical interpretation of the state space for the flavor fields is a controversial one and it is the object of still open discussions [2–5]. On the other hand, the discovery [3] that the Pontecorvo field mixing transformation is a non-unitarily implementable transformation rests on firm mathematical grounds, so that it cannot be ignored in any discussion on the field mixing problems. In a recent paper [5] it has been thus considered the degree of arbitrariness involved in the construction of the flavor states starting from the results of refs. [3,4] (denoted by the authors of ref. [5] as the BV formalism).

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By using a more general mass parameterization than the one adopted in the BV formalism, the conclusion of the authors of ref. [5] has been that, since the mass arbitrariness shows up in the oscillation probability, the conclusions drawn in refs. [3] are unphysical. In particular, they conclude that the procedure adopted in ref. [3] by choosing a specific mass parameterization has no physical basis. They then examine the structure of the neutrino propagator. However, although the physical poles of the propagator are shown to coincide with the eigenvalues of the mass matrix in the Lagrangian and appear not to be affected by the mass parameterization arbitrariness, they are not able to escape the arbitrariness difficulty in the oscillation formula. Thus they are finally forced to admit that it is still necessary to investigate in detail how to derive oscillations formulas reflecting real experimental situations on the basis of the field theory.

Motivated by such a necessity, in the present paper we consider the arguments of ref. [5] and we show that, even by applying the more general mass parameterization there adopted, the mass arbitrariness disappears from the exact oscillation formula as derived in ref. [4]. Actually, the oscillation formula considered in ref. [5] is the approximate one which was derived in ref. [3], and not its exact form obtained in the Green's function formalism of ref. [4].

The conclusions of ref. [5] about the unphysical basis of the analysis or refs. [3,4] are thus ruled out. On the contrary, even though the problem of the interpretation of the flavor space may still be object of discussion, the field theoretical formalism derived in refs. [3,4] appears to be sound and, what is most interesting, it leads to the oscillation formula which is experimentally testable.

In the present paper we also shortly comment on the physical meaning of the mass arbitrariness which has been introduced by the authors of ref. [5] without explicit justification. These comments are also useful in order to clarify the physical meaning of the more general transformations used in ref. [5] and of the particular choice adopted in the BV formalism.

The paper is organized as follows. In Section II generalities on the formalism are reviewed, also considering the generalization of ref. [5]. In Section III the exact oscillation formula is rederived in the formalism of ref. [5] and it is shown to be independent of the mass arbitrariness. Section IV is devoted to further comments and conclusions.

II. MIXED FERMIONS

To be definite, let us consider an oversimplified model with two (Dirac) fermion fields with a mixed mass term:

$$\mathcal{L} = \bar{\nu}_e \left(i \not{\partial} - m_e\right) \nu_e + \bar{\nu}_\mu \left(i \not{\partial} - m_\mu\right) \nu_\mu - \; m_{e\mu} \; \left(\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e\right) \; . \tag{1}$$

The above Lagrangian is sufficient in order to describe the single-particle evolution of a mixed state, i.e. neutrino oscillations [4], and can be fully diagonalized by the transformation

$$\begin{aligned} \nu_e(x) &= \nu_1(x) \; \cos\theta + \nu_2(x) \; \sin\theta \\ \nu_\mu(x) &= -\nu_1(x) \; \sin\theta + \nu_2(x) \; \cos\theta \; , \end{aligned} \tag{2}$$

where θ is the mixing angle and $m_e = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta$, $m_{e\mu} = (m_2 - m_1) \sin\theta \cos\theta$. ν_1 and ν_2 therefore are non-interacting, free fields, anticommuting with each other at any space-time point. They are explicitly given by

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i} \alpha^r_{\mathbf{k},i}(t) \; + v^r_{-\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad i = 1, 2 \; . \tag{3}$$

with $\alpha^r_{\mathbf{k},i}(t) = e^{-i\omega_i t}\alpha^r_{\mathbf{k},i}(0)$, $\beta^r_{\mathbf{k},i}(t) = e^{-i\omega_i t}\beta^r_{\mathbf{k},i}(0)$ and $\omega_i = \sqrt{\mathbf{k}^2 + m_i^2}$. Here and in the following we use $t \equiv x_0$, when no misunderstanding arises. The vacuum for the α_i and β_i operators is denoted by $|0\rangle_{1,2}$: $\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0$. The anticommutation relations are the usual ones (see ref. [3]). The orthonormality and completeness relations are:

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \; , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = v^{r\dagger}_{-\mathbf{k},i} u^s_{\mathbf{k},i} = 0 \; , \quad \sum_r (u^r_{\mathbf{k},i} u^{r\dagger}_{\mathbf{k},i} + v^r_{-\mathbf{k},i} v^{r\dagger}_{-\mathbf{k},i}) = I \; . \tag{4}$$

The fields ν_e and ν_μ are thus completely determined through eq.(2), which can be rewritten in the following form (we use $(\sigma, j) = (e, 1), (\mu, 2)$):

$$\nu_\sigma(x) = G_\theta^{-1}(t)\, \nu_j(x)\, G_\theta(t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},j} \alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},j} \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5}$$

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right) \right] \tag{6}$$

where $G_\theta(t)$ is the generator of the mixing transformations (2) (see ref. [3] for a discussion of its properties).

Eq.(5) gives an expansion of the flavor fields ν_e and ν_μ in the same basis of ν_1 and ν_2. In the BV formalism the flavor annihilation operators are then identified with

$$\begin{pmatrix} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} = G_\theta^{-1}(t) \begin{pmatrix} \alpha^r_{\mathbf{k},j}(t) \\ \beta^{r\dagger}_{-\mathbf{k},j}(t) \end{pmatrix} G_\theta(t) \tag{7}$$

The BV flavor vacuum is defined as $|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)|0\rangle_{1,2}$.

The explicit expression of the flavor annihilation operators is (we choose $\mathbf{k} = (0, 0, |\mathbf{k}|)$):

$$\begin{pmatrix} \alpha^r_{\mathbf{k},e}(t) \\ \alpha^r_{\mathbf{k},\mu}(t) \\ \beta^{r\dagger}_{-\mathbf{k},e}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\mu}(t) \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta\, |U_\mathbf{k}| & 0 & s_\theta\, \epsilon^r\, |V_\mathbf{k}| \\ -s_\theta\, |U_\mathbf{k}| & c_\theta & s_\theta\, \epsilon^r\, |V_\mathbf{k}| & 0 \\ 0 & -s_\theta\, \epsilon^r\, |V_\mathbf{k}| & c_\theta & s_\theta\, |U_\mathbf{k}| \\ -s_\theta\, \epsilon^r\, |V_\mathbf{k}| & 0 & -s_\theta\, |U_\mathbf{k}| & c_\theta \end{pmatrix} \begin{pmatrix} \alpha^r_{\mathbf{k},1}(t) \\ \alpha^r_{\mathbf{k},2}(t) \\ \beta^{r\dagger}_{-\mathbf{k},1}(t) \\ \beta^{r\dagger}_{-\mathbf{k},2}(t) \end{pmatrix} \tag{8}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, $\epsilon^r \equiv (-1)^r$, and

$$|U_\mathbf{k}| \equiv u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1} = v^{r\dagger}_{-\mathbf{k},1} v^r_{-\mathbf{k},2} \tag{9}$$

$$|V_\mathbf{k}| \equiv \epsilon^r \; u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2} = -\epsilon^r \; u^{r\dagger}_{\mathbf{k},2} v^r_{-\mathbf{k},1} \; . \tag{10}$$

We have:

$$|U_\mathbf{k}| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{k}|^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right) \tag{11}$$

$$|V_\mathbf{k}| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(\frac{|\mathbf{k}|}{(\omega_{k,2} + m_2)} - \frac{|\mathbf{k}|}{(\omega_{k,1} + m_1)} \right) \tag{12}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1 \tag{13}$$

It has been recently noticed [5], however, that expanding the flavor fields in the same basis as the (free) fields with definite masses is actually a special choice, and that a more general possibility exists.

In other words, in the expansion eq.(5) one could use eigenfunctions with arbitrary masses μ_σ, and therefore not necessarily the same as the masses which appear in the Lagrangian. On this basis, the authors of ref. [5] have generalized the BV transformation (7) by writing the flavor fields as

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},\sigma} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},\sigma} \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{14}$$

where u_σ and v_σ are the helicity eigenfunctions with mass μ_σ[1]. We denote by a tilde the generalized flavor operators introduced in ref. [5] in order to distinguish them from the ones defined in eq.(7). The expansion eq.(14) is more general than the one in eq.(5) since the latter corresponds to the particular choice $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$.

The relation between the flavor and the mass operators is now:

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} = K^{-1}_{\theta,\mu}(t) \begin{pmatrix} \alpha^r_{\mathbf{k},j}(t) \\ \beta^{r\dagger}_{-\mathbf{k},j}(t) \end{pmatrix} K_{\theta,\mu}(t) \quad , \tag{15}$$

with $(\sigma, j) = (e, 1), (\mu, 2)$ and where $K_{\theta,\mu}(t)$ is the generator of the transformation (2) and can be written as

$$K_{\theta,\mu}(t) = I_\mu(t)\, G_\theta(t) \tag{16}$$

$$I_\mu(t) = \prod_{\mathbf{k},r} \exp \left\{ i \sum_{(\sigma,j)} \xi^{\mathbf{k}}_{\sigma,j} \left[\alpha^{r\dagger}_{\mathbf{k},j}(t) \beta^{r\dagger}_{-\mathbf{k},j}(t) + \beta^r_{-\mathbf{k},j}(t) \alpha^r_{\mathbf{k},j}(t) \right] \right\} \tag{17}$$

with $\xi^{\mathbf{k}}_{\sigma,j} \equiv (\chi_\sigma - \chi_j)/2$ and $\cot\chi_\sigma = |\mathbf{k}|/\mu_\sigma$, $\cot\chi_j = |\mathbf{k}|/m_j$. For $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$ one has $I_\mu(t) = 1$.

The explicit matrix form of the flavor operators is [5]:

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},e}(t) \\ \widetilde{\alpha}^r_{\mathbf{k},\mu}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},e}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\mu}(t) \end{pmatrix} = \begin{pmatrix} c_\theta\, \rho^{\mathbf{k}}_{e1} & s_\theta\, \rho^{\mathbf{k}}_{e2} & ic_\theta\, \lambda^{\mathbf{k}}_{e1} & is_\theta\, \lambda^{\mathbf{k}}_{e2} \\ -s_\theta\, \rho^{\mathbf{k}}_{\mu 1} & c_\theta\, \rho^{\mathbf{k}}_{\mu 2} & -is_\theta\, \lambda^{\mathbf{k}}_{\mu 1} & ic_\theta\, \lambda^{\mathbf{k}}_{\mu 2} \\ ic_\theta\, \lambda^{\mathbf{k}}_{e1} & is_\theta\, \lambda^{\mathbf{k}}_{e2} & c_\theta\, \rho^{\mathbf{k}}_{e1} & s_\theta\, \rho^{\mathbf{k}}_{e2} \\ -is_\theta\, \lambda^{\mathbf{k}}_{\mu 1} & ic_\theta\, \lambda^{\mathbf{k}}_{\mu 2} & -s_\theta\, \rho^{\mathbf{k}}_{\mu 1} & c_\theta\, \rho^{\mathbf{k}}_{\mu 2} \end{pmatrix} \begin{pmatrix} \alpha^r_{\mathbf{k},1}(t) \\ \alpha^r_{\mathbf{k},2}(t) \\ \beta^{r\dagger}_{-\mathbf{k},1}(t) \\ \beta^{r\dagger}_{-\mathbf{k},2}(t) \end{pmatrix} \tag{18}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$ and

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[1]The use of such a basis simplifies considerably calculations with respect to the original choice of ref. [3].

$$\rho^{\mathbf{k}}_{ab}\delta_{rs} \equiv \cos\frac{\chi_a-\chi_b}{2}\delta_{rs} = u^{r\dagger}_{\mathbf{k},a}u^{s}_{\mathbf{k},b} = v^{r\dagger}_{-\mathbf{k},a}v^{s}_{-\mathbf{k},b} \tag{19}$$

$$i\lambda^{\mathbf{k}}_{ab}\delta_{rs} \equiv i\sin\frac{\chi_a-\chi_b}{2}\delta_{rs} = u^{r\dagger}_{\mathbf{k},a}v^{s}_{-\mathbf{k},b} = v^{r\dagger}_{-\mathbf{k},a}u^{s}_{\mathbf{k},b} \tag{20}$$

with $a,b = 1,2,e,\mu$. Since $\rho^{\mathbf{k}}_{12} = |U_{\mathbf{k}}|$ and $i\lambda^{\mathbf{k}}_{12} = \epsilon^r|V_{\mathbf{k}}|$, etc., the operators (18) reduce to the ones in eqs.(8) when $\mu_e \equiv m_1$ and $\mu_\mu \equiv m_2$[2].

The generalization of the BV flavor vacuum, which is annihilated by the flavor operators given by eq.(15), is now written as [5]:

$$|\widetilde{0}(t)\rangle_{e,\mu} \equiv K^{-1}_{\theta,\mu}(t)|0\rangle_{1,2} \quad . \tag{21}$$

For $\mu_e \equiv m_1$ and $\mu_\mu \equiv m_2$, this state reduces to the BV flavor vacuum $|0(t)\rangle_{e,\mu}$ above defined.

For the considerations which follow, it is also useful to report here the relation, given in ref. [5], between the general flavor operators of eq.(15) and the BV ones:

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} = J^{-1}_\mu(t)\begin{pmatrix} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} J_\mu(t) \quad , \tag{22}$$

$$J_\mu(t) = \prod_{\mathbf{k},r}\exp\left\{ i\sum_{(\sigma,j)}\xi^{\mathbf{k}}_{\sigma,j}\left[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) + \beta^{r}_{-\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t)\right]\right\} . \tag{23}$$

III. THE OSCILLATION FORMULA

In the formal framework of the previous Section the annihilation operators and the vacuum for mixed fermions are defined self-consistently, and the Hilbert space for mixed neutrinos can thus be constructed. Such an Hilbert space, however, has built in the arbitrariness related with the mass parameters μ_σ, $\sigma = e,\mu$. According to the authors of ref. [5], such an arbitrariness also shows up in the final expression of the oscillation probability, which is a non-acceptable result since the theory arbitrary parameters should have no effects on observable quantities. The full construction, although mathematically consistent, would then be questionable from a physical point of view. Now we show that the analysis of ref. [5] is not complete, that the exact oscillation probabilities are independent of the arbitrary mass parameters and therefore the conclusion of ref. [5] is ruled out.

The main point is that the authors of ref. [5] miss to compute the full oscillation probability whose exact form is presented in ref. [4] and it is there obtained in the Green's function formalism. In fact, the statement of ref. [5] that the oscillation formula "seems not to be correct" since it is based on the one neutrino state, which does depend on μ_e and μ_μ, is not correct: as we show below, it is possible to calculate the oscillation probabilities by using the

[2]In performing such an identification, one should take into account that the operators for antiparticles differ for a minus sign, related to the different spinor bases used in the expansions (5) and (14). Such a sign difference is however irrelevant in what follows.

arbitrary mass formalism of Section II, getting a result which is independent of the arbitrary masses μ_e and μ_μ and coincides with the one of ref. [4].

In the line of ref. [4], let us consider the propagator for the flavor fields, which has to be defined on the proper (flavor) vacuum. Notice that here we perform the computations in the generalized BV formalism of ref. [5]. The propagators are then given by:

$$\begin{pmatrix} \widetilde{G}_{ee}(x,y) & \widetilde{G}_{\mu e}(x,y) \\ \widetilde{G}_{e\mu}(x,y) & \widetilde{G}_{\mu\mu}(x,y) \end{pmatrix} \equiv {}_{e,\mu}\langle\widetilde{0}(y_0)| \begin{pmatrix} T\left[\nu_e(x)\bar{\nu}_e(y)\right] & T\left[\nu_\mu(x)\bar{\nu}_e(y)\right] \\ T\left[\nu_e(x)\bar{\nu}_\mu(y)\right] & T\left[\nu_\mu(x)\bar{\nu}_\mu(y)\right] \end{pmatrix} |\widetilde{0}(y_0)\rangle_{e,\mu}\,, \tag{24}$$

where the state used is the one defined in eq.(21). These propagators clearly *do depend* on the arbitrary parameters μ_e and μ_μ, which are present in $|\widetilde{0}(y_0)\rangle_{e,\mu}$. However, the propagator is not a measurable quantity: on the contrary, the oscillation probability, which can be defined in terms of it, is measurable and should not be affected by any arbitrary parameters.

Let us then consider the case of an initial electron neutrino which evolves (oscillates) in time. The two relevant propagators are [4]:

$$i\widetilde{G}^{>}_{ee}(t,\mathbf{x};0,\mathbf{y}) = {}_{e,\mu}\langle\widetilde{0}|\nu_e(t,\mathbf{x})\;\bar{\nu}_e(0,\mathbf{y})|\widetilde{0}\rangle_{e,\mu} \tag{25}$$

$$i\widetilde{G}^{>}_{\mu e}(t,\mathbf{x};0,\mathbf{y}) = {}_{e,\mu}\langle\widetilde{0}|\nu_\mu(t,\mathbf{x})\;\bar{\nu}_e(0,\mathbf{y})|\widetilde{0}\rangle_{e,\mu} \tag{26}$$

where $|\widetilde{0}\rangle_{e,\mu} \equiv |\widetilde{0}(t=0)\rangle_{e,\mu}$. As discussed in [4], there are four distinct transition amplitudes which can be defined from the above propagators. In the present generalized case, we have to use the wave functions u_σ and v_σ, instead of u_j and v_j which were used in [4]. This is in line with the discussion of the previous Section: the choice of the basis in which we expand the flavor fields determines the relevant annihilators and then the vacuum.

The amplitudes are then obtained as[3]

$$\widetilde{\mathcal{P}}^r_{ee}(\mathbf{k},t) \equiv i\,u^{r\dagger}_{\mathbf{k},e}\,\widetilde{G}^{>}_{ee}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},e} = \left\{\widetilde{\alpha}^r_{\mathbf{k},e}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\} \tag{27}$$

$$\widetilde{\mathcal{P}}^r_{\bar{e}e}(\mathbf{k},t) \equiv i\,v^{r\dagger}_{-\mathbf{k},e}\,\widetilde{G}^{>}_{ee}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},e} = \left\{\widetilde{\beta}^{r\dagger}_{-\mathbf{k},e}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\} \tag{28}$$

$$\widetilde{\mathcal{P}}^r_{\mu e}(\mathbf{k},t) \equiv i\,u^{r\dagger}_{\mathbf{k},\mu}\,\widetilde{G}^{>}_{\mu e}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},e} = \left\{\widetilde{\alpha}^r_{\mathbf{k},\mu}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\} \tag{29}$$

$$\widetilde{\mathcal{P}}^r_{\bar{\mu}e}(\mathbf{k},t) \equiv i\,v^{r\dagger}_{-\mathbf{k},\mu}\,\widetilde{G}^{>}_{\mu e}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},e} = \left\{\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\mu}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\} \tag{30}$$

The explicit form of these amplitudes is rather complicated. Notice that all of them involve the arbitrary parameters μ_σ. However, it can be verified that the following sum rule for the squared moduli is still valid:

$$\left|\widetilde{\mathcal{P}}^r_{ee}(\mathbf{k},t)\right|^2 + \left|\widetilde{\mathcal{P}}^r_{\bar{e}e}(\mathbf{k},t)\right|^2 + \left|\widetilde{\mathcal{P}}^r_{\mu e}(\mathbf{k},t)\right|^2 + \left|\widetilde{\mathcal{P}}^r_{\bar{\mu}e}(\mathbf{k},t)\right|^2 = 1\,, \tag{31}$$

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[3]With respect to the ones defined in ref. [4], we omit here an (irrelevant) phase factor. This is due to the different definition of the flavor operators – see eq.(8).

Moreover, through somewhat long direct calculation or by employing the linear relation eq.(22) (cf. also the second relation in (2.31) of ref. [5]) as well as the charge conjugation relation between $\alpha^r_{\mathbf{k},\sigma}$ and $\beta^r_{-\mathbf{k},\sigma}$, we obtain

$$\left|\left\{\widetilde{\alpha}^r_{\mathbf{k},e}(t),\widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\widetilde{\beta}^{r\dagger}_{-\mathbf{k},e}(t),\widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 = \left|\left\{\alpha^r_{\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \tag{32}$$

$$\left|\left\{\widetilde{\alpha}^r_{\mathbf{k},\mu}(t),\widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\mu}(t),\widetilde{\alpha}^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 = \left|\left\{\alpha^r_{\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \tag{33}$$

which is the announced result. In fact in ref. [4] the probabilities for oscillating neutrinos were found to be

$$\begin{aligned} P_{\nu_e\to\nu_e}(\mathbf{k},t) &= \left|\left\{\alpha^r_{\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \qquad (34)\\ &= 1-\sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right], \\ P_{\nu_e\to\nu_\mu}(\mathbf{k},t) &= \left|\left\{\alpha^r_{\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \qquad (35)\\ &= \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right]. \end{aligned}$$

Thus, we have proven that the generalized formalism of ref. [5] leads to the same exact result of ref. [4]. The above formula is still valid in the more general case: the probabilities for oscillating neutrinos do not depend on any arbitrary mass parameters.

IV. COMMENTS AND CONCLUSION

Although already discussed in detail in ref. [4], some comments about the oscillation formulas (34),(35) may be useful, in order to better clarify why the cancellation of the unphysical parameters occurs in eqs.(32),(33). In order to make clearer the physical reasoning of adopting the equal time vacua in the computation of the amplitudes, we remark that the quantities in eqs.(34),(35) are nothing but the expectation values (on the electron neutrino state at time t) of the charge operators $Q_\sigma \equiv \alpha^\dagger_\sigma\alpha_\sigma - \beta^\dagger_\sigma\beta_\sigma$ ($\sigma = e,\mu$ and we have suppressed the momentum and spin indices for simplicity). We have indeed

$$\mathcal{Q}_\sigma(t) \equiv \langle\nu_e(t)|Q_\sigma|\nu_e(t)\rangle = \left|\left\{\alpha_\sigma(t),\alpha^\dagger_e(0)\right\}\right|^2 + \left|\left\{\beta^\dagger_\sigma(t),\alpha^\dagger_e(0)\right\}\right|^2, \tag{36}$$

$$_{e,\mu}\langle 0(t)|Q_\sigma|0(t)\rangle_{e,\mu} = 0 \quad , \quad \langle\nu_e(t)|\left(Q_e + Q_\mu\right)|\nu_e(t)\rangle = 1, \tag{37}$$

where $|\nu_e(t)\rangle \equiv \exp\left[-iHt\right]\ \alpha^\dagger_e|0\rangle_{e,\mu}$.

In this way, the physically obvious fact is confirmed, that the measure of the flavor oscillation probabilities at time t (eqs.(34),(35)) is given by the expectation value of the flavor charges, $\mathcal{Q}_\sigma(t)$. On the other hand, the already established result of ref. [4], by which

the Green's functions eq.(24) are well defined because of the use of the equal time vacua, also confirms the above physical picture from a more formal point of view, and it is strictly related to it. It has been shown in ref. [4] that quantities like ${}_{e\mu}\langle 0(x^0)|\nu_\sigma(x)\bar{\nu}_e(0,\vec{y})|0(0)\rangle_{e\mu}$ simply vanish (in the infinite volume limit), due to the unitary inequivalence of flavor vacua at different times (we also notice that at finite volume such a quantity does depend on the arbitrary parameters above introduced).

We further observe that eq.(37) simply and consistently expresses the conservation of the total charge. It is remarkable that, according to the analysis performed in ref. [3], the operator for the total charge $Q_e + Q_\mu$ is the Casimir operator for the $su(2)$ algebra associated with the mixing transformations eq.(2), and consequently it commutes with the mixing generator (6) (and (16)).

Finally, for the full understanding of the result (32),(33), it is essential to remark that the charge operators Q_σ are *invariant* under the action of the Bogoliubov generator eq.(23), i.e. $\widetilde{Q}_\sigma = Q_\sigma$, where $\widetilde{Q}_\sigma \equiv \widetilde{\alpha}^\dagger_\sigma\widetilde{\alpha}_\sigma - \widetilde{\beta}^\dagger_\sigma\widetilde{\beta}_\sigma$. Besides the direct computations leading to eqs.(32),(33), such an invariance, together with eq.(36), provides a strong and immediate proof of the independence of the oscillation formula from the μ_σ parameters.

In ref. [5] the flavor field wave functions u_σ and v_σ have been introduced which satisfy the free Dirac equations with arbitrary mass μ_σ ($\sigma = e, \mu$). Its introduction has not been justified in ref. [5]. Therefore a short comment about the physical meaning of such a procedure may be in order and it can be also useful for a better understanding of the formalism. Use of the wave functions u_σ and v_σ clearly represents a more general choice than the one made in the BV formalism [3], where $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$ has been used.

We observe that the mass parameter μ_σ represents the "bare" mass of the corresponding field and therefore it can be given any arbitrary value. Moreover, for $\theta = 0$ the transformation (15) reduces to the transformation generated by $I_\mu(t)$ given by eq.(17): now note that this is nothing but a Bogoliubov transformation which, at $\theta = 0$, relates unmixed field operators, α_j and, say, $a_j(\xi_{\sigma,j})$, of masses m_j and μ_σ, respectively. In the language of the LSZ formalism of QFT [7,8], the α_j refer to physical (free) fields and the $a_j(\xi_{\sigma,j})$ to Heisenberg (interacting) fields. In the infinite volume limit, the Hilbert spaces where the operators α_j and a_j are respectively defined, turn out to be unitarily inequivalent spaces. Moreover, the transformation parameter $\xi_{\sigma,j}$ acts as a label specifying Hilbert spaces unitarily inequivalent among themselves (for each (different) value of the μ_σ mass parameter). The crucial point is that the physically relevant space is the one associated with the observable physical mass m_j, the other ones being associated with the bare masses μ_σ. It can be shown [7] that the masses μ_σ dynamically acquire a convenient mass shift term such that the asymptotic physical α_j-fields are associated with the physical mass m_j and the arbitrariness intrinsic to the bare mass μ_σ does not affect the observables.

Therefore, in principle any one of the ξ-parameterized Hilbert spaces can be chosen to work with (in other words, the bare masses can be given any arbitrary value). Since, however, one is interested in observable quantities, in the LSZ formalism [7,8] the space one chooses to work with is the free physical field space (associated to the α_j operator fields, in our case). This is the "particular" choice made in the BV formalism. In the generalized BV formalism instead, by means of the Bogoliubov transformation explicitly given by eq.(18) written for $\theta = 0$, one first moves to the operators $a_j(\xi_{\sigma,j})$, leaving the ξ value unspecified (i.e. for arbitrary mass parameter μ_σ) and then one considers the mixing problem. Of course,

at the end of the computations observable quantities should not depend on the arbitrary parameters, as indeed in this paper we have proven it happens to be.

Here we are not going to give more details on the multiplicity of Hilbert spaces associated with arbitrary bare mass parameters. However, the above comment sheds some light on the physical meaning of the particular choice made in refs. [3,4], and it also suggests to us why the result of the computations presented in the present paper actually was to be expected on a physical ground, besides being supported by straightforward mathematics.

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